Holomorphic functions associated with indeterminate rational moment problems☆

A. Bultheel[a,*], E. Hendriksen[b], O. Njåstad[c]

[a] Dept. Computer Sc., KU Leuven, Celestijnenlaan 200A, 3001 Heverlee, Belgium
[b] Beerstratenlaan 23, 2421GN Nieuwkoop, The Netherlands
[c] Dept. of Mathematical Sciences, Norwegian Univ. of Science and Technology, Trondheim, Norway

Abstract

We consider indeterminate rational moment problems on the real line with their associated orthogonal rational functions. There exists a Nevanlinna type parameterization relating to the problem, with associated Nevanlinna matrices of functions having singularities in the closure of the set of poles of the rational functions belonging to the problem. We prove that the growth at the essential singularities of the four functions in the Nevanlinna matrix is the same.

Keywords: rational moment problems, orthogonal rational functions, Nevanlinna parameterization, Riesz type criterion, growth estimates
2000 MSC: 30D15, 30E05, 42C05, 44A60

1. Motivation

To compute an integral $I_\mu(f) = \int_{\mathbb{R}} f(x)\mathrm{d}\mu(x)$ where μ is a nonnegative measure, one can make use of Gauss-type quadrature formulas. These compute an approximation $I_n(f) = \sum_{k=1}^n w_{nk} f(x_{nk})$ with positive weights w_{nk} and whose nodes x_{nk} are the zeros of the nth degree polynomial ϕ_n orthogonal with respect to the measure μ. Since a node x_{nk} could coincide with a pole of the function f, it is safer to introduce a parameter $\tau_n \in \mathbb{R}$ and use the zeros of a quasi-orthogonal polynomials $\phi_n + \tau_n\phi_{n-1}$. This τ_n can then be used to move the zeros away from the poles of f. It is well known that the Gauss-type quadrature formulas based on the zeros of quasi-orthogonal polynomials will have algebraic degree of exactness $2n-2$ ($2n-1$ if $\tau_n = 0$). This means that it will match the power moments: $c_k = I_\mu(x^k) = I_n(x^k)$, $k = 0,1,\ldots,2n-2$. This links the study of convergence of these quadrature formulas to the solution of a Hamburger moment problem in the sense that the discrete measure of the quadrature $\mu_n(x) = \sum_k w_{nk}\delta(x - x_{nk})$ will (hopefully) converge (in a weak sense) to a solution of the moment problem. In fact, for all functions f in a class in which the polynomials are dense the integral $I_\nu(f)$ will be the same for any ν that solves the moment problem, that is any nonnegative measure ν on $\mathbb{R}$ that satisfies $I_\nu(x^k) = I_\mu(x^k)$, $k = 0,1,2,\ldots$. Thus convergence of the quadrature does not imply that μ_n will converge to μ in a stronger sense. Indeed, when the moment problem is indeterminate there are infinitely many solutions of the moment problem and the μ from the original integral is just one of them. For example when $\mathrm{d}\mu(x) = \exp\{-\sqrt{x}\}\mathrm{d}x$, then, according to [3, Thm. 4.1], the moment problem will be indeterminate. All solutions of the moment problem can be characterized via their Stieltjes transform by the so called Nevanlinna parametrization (more details in the next section)

$$S_\mu(z) = -\frac{a(z)f(z) - c(z)}{b(z)f(z) - d(z)}$$

with f an arbitrary Pick function. The four functions a,b,c,d holomorphic in $\mathbb{C}\setminus\mathbb{R}$ appear in a 2×2 Nevanlinna matrix and they have zeros that are all in the support of particular extreme solutions of the moment problem (corresponding

☆To the memory of Pablo González-Vera a colleague and dear friend

*Corresponding author
Email addresses: adhemar.bultheel@cs.kuleuven.be (A. Bultheel), erik.hendriksen@planet.nl (E. Hendriksen), njastad@math.ntnu.no (O. Njåstad)

to the choice $f = 0$ and $f = \infty$). Since these functions are obtained as the limits of quasi-orthogonal polynomials, the distribution of their zeros will say something about the asymptotic behaviour of the zeros of these quasi-orthogonal polynomials, i.e., the nodes of the corresponding quadratures.

Now suppose we want to compute

$$I_\mu(f) = \int_{\mathbb{R}} f(x)\mathrm{d}\mu(x), \quad f(x) = \sin(|x-5|^{-1})\log\left(\left|\frac{x-0.5}{x+0.2}\right|\right), \ \mathrm{d}\mu(x) = \mathrm{e}^{-\sqrt{x}}\mathrm{d}x.$$

In that case $f(x)$ has three essential singularities $x = 5$, $x = 0.5$ and $x = -0.2$. Therefore it is much wiser to replace the set of orthogonal polynomials with the set of orthogonal rational functions of the form $p_n(x)/\pi_n(x)$ where p_n is a polynomial of degree n at most and $\pi_n(x) = \prod_{k=1}^n (1 - x/\alpha_k)$ where in this example we can choose $\alpha_k = 5$ for $k = 3l$, $\alpha_k = 0.5$ for $k = 3l-1$ and $\alpha_k = -0.2$ for $k = 3l-2$ where $l = 1,2,\ldots$. Such a construction can be set up in any situation where a finite number of different real poles $\Gamma = \{\alpha_k : k = 1,\ldots,m\}$ are used in the orthogonal rational function spaces and each of them is repeated an infinite number of times. The rational versions of the Gauss quadrature formulas will now fit rational moments, i.e., $I_\mu(b_k) = I_n(b_k)$ for $k = 0,\ldots,2n-2$ where $b_0 = 1$ and $b_k(x) = x^k/\pi_k(x)$, $k = 1,2,\ldots$. So this links the rational quadrature formulas with rational Hamburger moment problems. As in the polynomial case there holds a Nevanlinna parametrization to characterize all the solutions of the rational moment problem. The four functions in the Nevanlinna matrix are limits of quasi-orthogonal rational functions and their zeros belong to the supports of either one of the two extreme solutions of the moment problem and they will get a positive mass. These zeros will accumulate at the points $\alpha_k \in \Gamma$ that will thus belong to the support of both of the extreme solutions but with zero mass.

With this paper we continue our investigation of the behaviour of these Nevanlinna functions in the rational case that we started in [12]. We shall give more details on how these functions behave at the points $\alpha_k \in \Gamma$. In fact the behaviour will be the same for all four functions. This extends results of Berg and Pedersen [4] obtained in the polynomial case. To prove this we will also show that the way in which their zeros accumulate at the $\alpha_k \in \Gamma$ is the same for all four functions. Before we engage in the formulation and the proofs of our main results, we will first recall some definitions and results from the literature.

2. Definitions and notation

We use the following notations. $\mathbb{C}$ denotes the complex plane, $\hat{\mathbb{C}}$ the extended complex plane (one point compactification), $\mathbb{R}$ the real line, $\hat{\mathbb{R}}$ the closure of $\mathbb{R}$ in $\hat{\mathbb{C}}$, $\mathbb{U}$ the open upper half-plane, $\hat{\mathbb{U}}$ the closure of $\mathbb{U}$ in $\hat{\mathbb{C}}$.

A function f is called a *Pick function* if it is holomorphic in $\mathbb{U}$ and maps $\mathbb{U}$ into $\hat{\mathbb{U}}$. A Pick function is either a constant in $\hat{\mathbb{R}}$ or maps $\mathbb{U}$ into $\mathbb{U}$.

We define the integral transformations Ω_μ and S_μ of a finite measure on $\mathbb{R}$ by

$$\Omega_\mu(z) = \int_{\mathbb{R}} \frac{1+tz}{t-z}\mathrm{d}\mu(t) \ \text{ and } \ S_\mu(z) = \int_{\mathbb{R}} \frac{1}{t-z}\mathrm{d}\mu(t). \tag{2.1}$$

The functions Ω_μ and S_μ are Pick functions and they satisfy

$$\Omega_\mu(z) = (1+z^2)S_\mu(z) + z\int_{\mathbb{R}} \mathrm{d}\mu(t). \tag{2.2}$$

Let M be a Hermitian, positive definite linear functional on the space $\mathscr{P}$ of polynomials, and define its moments c_n by $c_n = M[z^n]$, $n = 0,1,2,\ldots$. A solution of the *Hamburger moment problem* for $\{c_n\}$ (or M) is a (positive) measure μ on $\mathbb{R}$ which satisfies $\int_{\mathbb{R}} t^n \mathrm{d}\mu(t) = c_n$ for all $n = 0,1,2,\ldots$.

A moment problem is called *determinate* if it has exactly one solution, *indeterminate* if it has more than one solution.

H. Hamburger (in [15–17]) showed that such measures exist, and gave conditions for the moment problem to be determinate (i.e., to have a unique solution).

R. Nevanlinna (see [21, 22]) established a one-to-one correspondence between all Pick functions f and all solutions μ of an indeterminate moment problem, given by

$$S_\mu(z) = -\frac{a(z)f(z) - c(z)}{b(z)f(z) - d(z)}.$$

(*Nevanlinna parameterization* of the solutions.) Here a,b,c,d are certain entire transcendental functions. It was shown by *M. Riesz* (see [26–28] and also [1, Ch. 3]) that the growth of these functions are restricted as follows: For every positive constant ε, there exists a constant $M(\varepsilon)$ such that

$$|F(z)| \leq M(\varepsilon)\exp\{\varepsilon|z|\},$$

where F is any of the functions a,b,c,d. Thus these function are of order less than one, or of zero type of order one.

In [4] it was shown by *Berg and Pedersen* that the order (and the type) are always the same for the functions a,b,c,d, for a given indeterminate problem.

A parameterization of the solutions can also be given in terms of *Pick functions g* and the integral transforms Ω_μ through the formula

$$\Omega_\mu(z) = -\frac{A(z)g(z)-C(z)}{B(z)g(z)-D(z)},$$

where A,B,C,D are certain entire transcendental functions with simple relationships to the functions a,b,c,d. The functions A,B,C,D satisfy the same condition for restriction on the growth as the functions a,b,c,d do.

For more details and further results concerning the Nevanlinna parametrization we refer to [3, 6, 14, 29, 30] in addition to the references already cited.

In this paper we treat a *rational moment problem*, where the polynomials are replaced by rational functions with poles in $\hat{\mathbb{R}}$. A Nevanlinna-type parametrization for solutions of an indeterminate rational problem in terms of Pick functions, the integral transforms Ω_μ and certain holomorphic functions A,B,C,D was proved by *Almendral* in [2]. In [12], *Bultheel, González-Vera, Hendriksen and Njåstad* treated especially the situation where the set of singularities for the rational functions is finite, with poles of all orders occurring. Maximal estimates of the growth of the functions A,B,C,D in the parametrization formula at the singularities were established, analogous to those for the classical problem. Our aim in this paper is to prove that at each singularity the order of growth of A,B,C,D are equal.

Properties of solutions of *strong (or two-point) Hamburger moment problems* (where the singularities alternate between the origin and infinity) were treated e.g. in [18, 19, 23–25].

A parametrization result for an indeterminate rational moment problem where the singularities are contained in the open unit disk (or equivalently in the open upper half plane) was established in [11].

The outline of the paper is as follows. In Section 3 we introduce the rational moment problem and the associated quadrature formulas that will play an essential role in its solution. Section 4 gives the Nevanlinna parametrization of the solutions of indeterminate problem. In Section 5 we discuss the zeros and the properties of the functions A,B,C,D. These are used in Section 6 to give a factorization of these functions. Finally in Section 7 we prove our result on the equality of the growth orders.

3. A rational moment problem

We shall here consider a somewhat special case of rational moment problems. For treatment of general problems, we refer to [7–12].

Let $\{\alpha_k\}_{k=1}^\infty$ be a sequence of arbitrary points (singularities or interpolation points) in $\hat{\mathbb{R}}\setminus\{0\}$ and set $\alpha_0=\infty$. We set

$$\Gamma = \{\alpha \in \hat{\mathbb{R}} : \text{There exists an } n \text{ such that } \alpha_n = \alpha\}.$$

For $\alpha\in\Gamma$, we denote by Γ_α the subsequence of those α_{n_k} in $\{\alpha_n\}$ for which $\alpha_{n_k}=\alpha$. *We shall here assume that* Γ *is finite and that every* Γ_α *is infinite.* We may write $\Gamma=\{\gamma_1,\gamma_2,\dots,\gamma_q\}$.

We set

$$\begin{aligned}\pi_0 = 1, \quad \pi_n(z) &= \prod_{k=1}^n\left(1-\frac{z}{\alpha_k}\right), \quad n=1,2,\dots, \\ b_n(z) &= \frac{z^n}{\pi_n(z)}, \quad n=0,1,2,\dots.\end{aligned} \tag{3.1}$$

Note that $\overline{b}_n = b_n$, thus $b_n(x)$ is real for real x. The set $\{b_0, b_1, \dots, b_n\}$ is a basis for the space

$$\mathscr{L}_n = \left\{ \frac{p(z)}{\pi_n(z)} : p \in \mathscr{P}_n \right\}$$

where $\mathscr{P}_n$ denotes the space of polynomials of degree at most n. We define $\mathscr{L} = \bigcup_{n=0}^{\infty} \mathscr{L}_n$. Thus $\mathscr{L}$ consists of all rational functions L of the form $L(z) = \frac{p(z)}{\pi_n(z)}$, $p \in \mathscr{P}_n$, for some $n = 0, 1, 2, \dots$. Note that $\mathscr{L} \cdot \mathscr{L} = \mathscr{L}$, since all Γ_α are infinite and Γ is finite.

The situation $\alpha_n = \infty$ for all n represents the classical case, where $\mathscr{L} = \mathscr{P}$. In many situations the point ∞ requires special consideration. To keep the presentation without such extra considerations we shall in the following assume that $\infty \notin \Gamma$, but our main results will be valid also when $\infty \in \Gamma$. In particular when $\Gamma = \{\infty\}$, the classical results are obtained. The reason for $0 \notin \Gamma$ is of a technical kind. The theory where every point in $\hat{\mathbb{R}}$ may occur in Γ becomes rather more complicated (cf. [10]).

Let M be a Hermitian, positive definite linear functional on $\mathscr{L}$. For convenience we assume M to be normalized such that $M[1] = 1$. The moments μ_n of M are defined as

$$\mu_n = M[b_n], \quad n = 0, 1, 2, \dots$$

A measure μ on $\mathbb{R}$ is said to solve the *rational Hamburger moment problem* for M if

$$\int_{\mathbb{R}} b_n(t) \mathrm{d}\mu(t) = \mu_n \quad \text{for} \quad n = 0, 1, 2, \dots. \tag{3.2}$$

or equivalently

$$\int_{\mathbb{R}} f(t) \mathrm{d}\mu(t) = M[f] \quad \text{for} \quad f \in \mathscr{L}. \tag{3.3}$$

We shall in the following be concerned mainly with *indeterminate moment problems*, i.e., problems where there is more than one measure satisfying (3.2) or (3.3).

Let $\{\varphi_n\}_{n=0}^{\infty}$ be the sequence of functions obtained by orthonormalization of the sequence $\{b_n\}_{n=0}^{\infty}$ with respect to the inner product $\langle \cdot, \cdot \rangle$ defined by $\langle f, g \rangle = M[f \cdot \overline{g}]$. We fix the elements uniquely by a unimodular factor such that the coefficient c_n of b_n in the expansion $\varphi_n = \sum_{k=0}^{n} c_k b_k$ is positive.

The function φ_n has the form $\varphi_n(z) = \frac{p_n(z)}{\pi_n(z)}$, $p_n \in \mathscr{P}_n \setminus \mathscr{P}_{n-1}$. We shall assume a *weak regularity condition*, namely $p_n(\alpha_{n-1}) \neq 0$ for all n.

The functions ψ_n of the second kind are defined as

$$\psi_0(z) = -z, \quad \psi_n(z) = M_t \left[\frac{1 + tz}{t - z} \{ \varphi_n(t) - \varphi_n(z) \} \right], \quad n = 1, 2, \dots.$$

where M_t means that M operates on the argument as a function of t. Equivalently

$$\psi_n(z) = \int_{\mathbb{R}} \frac{1 + tz}{t - z} \{ \varphi_n(t) - \varphi_n(z) \} \mathrm{d}\mu(t), \quad n = 1, 2, \dots$$

where μ is any solution of the moment problem. We observe that $\psi_n \in \mathscr{L}_n$ and that $\varphi_n(x)$ and $\psi_n(x)$ are real for real x.

Remark 3.1. We have here followed the convention used in [2] and [12]. The definition of ψ_n differs from the definition in the monograph [9], where the following convention is used:

$$\psi_0(z) = \mathrm{i}z, \quad \psi_n(z) = -\mathrm{i} M_t \left[\frac{1 + tz}{t - z} \{ \varphi_n(t) - \varphi_n(z) \} \right], \quad n = 1, 2, \dots.$$

Similarly in [9], the integral transformation Ω_μ is defined by $\Omega_\mu(z) = -\mathrm{i} \int_{\mathbb{R}} \frac{1+tz}{t-z} \mathrm{d}\mu(t)$.

A function of the form $\varphi_n(z) + \tau_n \frac{1-z/\alpha_{n-1}}{1-z/\alpha_n} \varphi_{n-1}(z)$ with $\tau_n \in \hat{\mathbb{R}}$ is called *quasi-orthogonal of order n*. (See [9, Ch. 11.5]). For convenience we shall extend this definition to functions of the form

$$a_n \varphi_n(z) + \tau_n \frac{1-z/\alpha_{n-1}}{1-z/\alpha_n} \varphi_{n-1}(z), \quad \tau_n \in \hat{\mathbb{R}}, \ a_n \in \mathbb{R},$$

so that also functions $\tau_n \frac{1-z/\alpha_{n-1}}{1-z/\alpha_n} \varphi_{n-1}(z)$ are counted as quasi-orthogonal of order n. A zero of the numerator of such a function could cancel a zero of its denominator. However, because of the interlacing property of the zeros of the numerators of successive ϕ_k, this can only happen for at most $n+1$ values of τ_n if we fix n and a_n. Therefore, except for a possibly at most countable set X of exceptional parameter values a_n, τ_n, every quasi-orthogonal function of order n has n simple real zeros when $a_n \neq 0$ and $n-1$ simple real zeros when $a_n = 0$ (see [9, Ch. 11.5]).

The zeros $\xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,n}$ (when $a_n \neq 0$) are nodes of a quadrature formula with positive weights $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,n}$ exact for functions in $\mathscr{L}_{n-1} \cdot \mathscr{L}_{n-1}$.
Similarly when $a_n = 0$, there is a quadrature formula with nodes at the $n-1$ zeros and positive weights which is exact on $\mathscr{L}_{n-2} \cdot \mathscr{L}_{n-1}$. The basic results concerning quasi-orthogonal rational functions and their associated quadrature formulas can be found in [9, Ch. 11.5–11.6].

Let ξ be an arbitrary fixed element in $\mathbb{R}$. We define the quasi-orthogonal function $\varphi_n(z, \xi)$ as

$$\varphi_n(z,\xi) = \frac{1-\xi/\alpha_{n-1}}{1-\xi/\alpha_n} \varphi_{n-1}(\xi) \varphi_n(z) - \frac{1-z/\alpha_{n-1}}{1-z/\alpha_n} \varphi_{n-1}(z) \varphi_n(\xi),$$

where ξ is chosen such that for $a_n = \frac{1-\xi/\alpha_n}{1-\xi/\alpha_{n-1}} \varphi_{n-1}(\xi)$ and $-\varphi_n(\xi)$ do not belong to the exceptional set X introduced above. Clearly ξ is a zero of $\varphi_n(z,\xi)$. It follows from the *determinant formula* (see e.g. [9, Ch. 11.3]) that two consecutive orthogonal functions $\varphi_{n-1}(z)$ and $\varphi_n(z)$ can not have a common zero. We shall number the zeros of $\varphi_n(z,\xi)$ such that $\xi = \xi_1$. For the associated weight $\lambda_{n,1}$ in the corresponding quadrature formula we shall write $\lambda_n(\xi)$. The formula then has the form

$$\int_{\mathbb{R}} f(z) \mathrm{d}\mu(z) \approx \lambda_n(\xi) f(\xi) + \sum_{k=2}^{n} \lambda_{n,k} f(\xi_{n,k}) \tag{3.4}$$

when $a_n \neq 0$ and analogously when $a_n = 0$.

The weights $\lambda_{n,k}$ can be expressed as $\lambda_{n,k} = \int_{\mathbb{R}} [L_{n,k}(t)]^2 \mathrm{d}\mu(t)$ where $L_{n,k}$ is the unique element in $\mathscr{L}_{n-1}$ for which $L_{n,k}(\xi_{n,j}) = \delta_{k,j}$, $k, j = 1, 2, \ldots, n$. In particular

$$\lambda_n(\xi) = \int_{\mathbb{R}} [L_{n,1}(t)]^2 \mathrm{d}\mu(t). \tag{3.5}$$

The value of the weight can also be expressed in the form $\lambda_{n,k} = 1/\sum_{j=0}^{n-1} [\varphi_j(\xi_{n,k})]^2$, $k = 1, 2, \ldots, n$. (See [9, Ch. 11.6].) In particular

$$\lambda_n(\xi) = \frac{1}{\sum_{j=0}^{n-1} [\varphi_j(\xi)]^2}. \tag{3.6}$$

Note that these concepts are independent of the solution μ and are only depending on the functional M.

We shall give arguments concerning the quasi-orthogonal functions $\varphi_n(z,\xi)$ that are strongly indebted to the analogous treatment in [14].

We shall use the notation $\mathscr{L}_n^R$ for the set of elements in $\mathscr{L}_n$ (or $\mathscr{L}_{n-1}$ if $a_n = 0$) where all the coefficients with respect to the basis $b_0, \ldots, b_n$ are real.

Proposition 3.2. *$\lambda_n(\xi)$ is characterized by*

$$\lambda_n(\xi) = \min \left\{ \int_{\mathbb{R}} [q_{n-1}(t)]^2 \mathrm{d}\mu(t) : q_{n-1} \in \mathscr{L}_{n-1}^R, \ q_{n-1}(\xi) = 1 \right\}$$

PROOF. Let $q_{n-1} \in \mathscr{L}^R_{n-1}$, $q_{n-1}(\xi) = 1$. First assume $\varphi_{n-1}(\xi) \neq 0$. Then by (3.4)

$$\int_{\mathbb{R}} [q_{n-1}(t)]^2 \mathrm{d}\mu(t) = \lambda_n(\xi)[q_{n-1}(\xi)]^2 + \sum_{k=2}^{n} \lambda_{n,k}[q_{n-1}(\xi_{n,k})]^2 \geq \lambda_n(\xi).$$

Next assume $\varphi_{n-1}(\xi) = 0$. Recall that then $\varphi_{n-2}(\xi) \neq 0$. By (3.6) we have $\lambda_n(\xi) = \lambda_{n-1}(\xi)$. We can write $q_{n-1}(z) = a\varphi_{n-1}(z) + q_{n-2}(z)$ with $q_{n-2} \in \mathscr{L}^R_{n-2}$. Note that $q_{n-2}(\xi) = 1$ since $\varphi_{n-1}(\xi) = 0$. Thus

$$\int_{\mathbb{R}} [q_{n-1}(t)]^2 \mathrm{d}\mu(t) = a^2 \int_{\mathbb{R}} [\varphi_{n-1}(t)]^2 \mathrm{d}\mu(t) + 2a \int_{\mathbb{R}} \varphi_{n-1}(t) q_{n-2}(t) \mathrm{d}\mu(t) + \int_{\mathbb{R}} [q_{n-2}(t)]^2 \mathrm{d}\mu(t).$$

The middle term vanishes by orthogonality. Hence $\int_{\mathbb{R}} [q_{n-1}(t)]^2 \mathrm{d}\mu(t) \geq \int_{\mathbb{R}} [q_{n-2}(t)]^2 \mathrm{d}\mu(t)$. By the first part of the proof, $\int_{\mathbb{R}} [q_{n-1}(t)]^2 \mathrm{d}\mu(t) \geq \lambda_{n-1}(\xi) = \lambda_n(\xi)$. Thus

$$\lambda_n(\xi) \leq \min \left\{ \int_{\mathbb{R}} [q_{n-1}(t)]^2 \mathrm{d}\mu(t) : \ q_{n-1}(\xi) \in \mathscr{L}^R_{n-1}, \ q_{n-1}(\xi) = 1 \right\}.$$

This together with (3.5) concludes the proof. $\square$

Clearly $\lambda_{n+1}(\xi) \leq \lambda_n(\xi)$. Thus the limit $\lambda(\xi) = \lim_n \lambda_n(\xi)$ exists. That also follows directly from (3.6).

Proposition 3.3. *Assume that a point $\alpha \in \Gamma$ does not belong to $\operatorname{supp}\mu$ for some solution μ of the rational moment problem. Then there exists a $\xi \in \mathbb{R} \setminus \Gamma$ in the neighborhood of α such that $\lambda(\xi) = 0$ with $\lambda(\xi)$ as defined above.*

PROOF. Set $d = \operatorname{dist}(\alpha, \operatorname{supp}\mu)$. Choose $\xi \in \mathbb{R} \setminus \Gamma$ such that $\frac{1-\xi/\alpha_n}{1-\xi/\alpha_{n-1}} \varphi_{n-1}(\xi)$ and $-\varphi_n(\xi)$ do not belong to the countable set X introduced above, for any n, and such that $\operatorname{dist}(\alpha, \xi) = rd$, $0 < r < 1$. There is for each m a smallest integer $n(m)$ such that

$$\left(\frac{1-\xi/\alpha}{1-z/\alpha} \right)^{m-1} \in \mathscr{L}_{n(m)-1}.$$

Set $q_{n(m)-1}(z) = \left(\frac{1-\xi/\alpha}{1-z/\alpha} \right)^{m-1}$. Then $q_{n(m)-1}(\xi) = 1$. For $t \in \operatorname{supp}\mu$ we have $|t-\alpha| \geq d$, while $|\xi - \alpha| = rd$. Consequently

$$\int_{\mathbb{R}} [q_{n(m)-1}(t)]^2 \mathrm{d}\mu(t) = \int_{\mathbb{R}} \left(\frac{\xi - \alpha}{t - \alpha} \right)^{2m-2} \mathrm{d}\mu(t) \leq \frac{r^{2m-2} d^{2m-2}}{d^{2m-2}} = r^{2m-2}.$$

This result together with Proposition 3.2 and the fact that $\lambda_{n(m)}(\xi) \leq \int_{\mathbb{R}} [q_{n(m)-1}(t)]^2 \mathrm{d}\mu(t)$ implies that $\lambda_{n(m)} \xrightarrow{m} 0$. Consequently $\lambda(\xi) = 0$. $\square$

4. Indeterminate problems and the functions A, B, C, D

We shall now concentrate on *indeterminate problems*. It is shown in [8] (where an equivalent setting with singularities on the unit circle is considered) that the moment problem is indeterminate if and only if the series $\sum_{k=0}^{\infty} |\varphi_k(z)|^2$ and $\sum_{k=0}^{\infty} |\psi_k(z)|^2$ converge for some $z \in \mathbb{U} \setminus \{\mathrm{i}\}$. See also [9, Ch. 11.7]. The *theorem of invariability* (see [8], [9, Ch. 11.7]) states that in this case, these series converge locally uniformly in $\mathbb{C} \setminus (\mathbb{R} \cup \{\mathrm{i}\} \cup \{-\mathrm{i}\})$. Analysis of the argument shows that there is locally uniform convergence in $\hat{\mathbb{C}} \setminus \Gamma$. In other words: when the problem is indeterminate, the series $\sum_{k=0}^{\infty} |\varphi_k(z)|^2$ and $\sum_{k=0}^{\infty} |\psi_k(z)|^2$ converge locally uniformly in $\hat{\mathbb{C}} \setminus \Gamma$. On the other hand, when the problem is determinate, the series $\sum_{k=0}^{\infty} |\varphi_k(z)|^2$ and $\sum_{k=0}^{\infty} |\psi_k(z)|^2$ diverge for every $z \in \mathbb{C} \setminus \mathbb{R}$.

From the considerations above we obtain the following necessary condition for a problem to be indeterminate

Theorem 4.1. *If the rational moment problem is indeterminate, then $\Gamma \subset \operatorname{supp}\mu$ for every solution μ.*

PROOF. Assume that $\alpha \notin \operatorname{supp}\mu$ for some $\alpha \in \Gamma$ and some solution μ. Then by Proposition 3.3 we have $\lambda(\xi) = 0$ for some $\xi \in \mathbb{R} \setminus \Gamma$. It follows then from (3.6) that the series $\sum_{k=0}^{\infty} |\varphi_k(\xi)|^2$ diverges. This means according to the discussion above that the problem is determinate. This contradiction proves the result. $\square$

Let x_0 be a fixed point in $\mathbb{R}$, $x_0 \notin \Gamma$, $x_0 \neq 0$. For technical reasons, x_0 is chosen such that $\psi_n(x_0) \neq 0$ and for all n, $q_n(\alpha_k, x_0) \neq 0$ for $k = 1, 2, \ldots, n$, where $q_n(z, \tau)$ is the numerator polynomial in the quasi-orthogonal rational function $\varphi_n(z) + \tau \frac{1-z/\alpha_{n-1}}{1-z/\alpha_n} \varphi_{n-1}(z)$. Such choice is always possible, see [9, Ch. 11.5].

We set $f_n(z,w) = (1 - z/\alpha_n)(1 - w/\alpha_{n-1})$ and define functions $A_n(z) = A_n(z,x_0)$, $B_n(z) = B_n(z,x_0)$, $C_n(z) = C_n(z,x_0)$, $D_n(z) = D_n(z,x_0)$ by

$$\begin{aligned} A_n(z) &= \frac{1}{E_n}[f_n(x_0,z)\psi_n(x_0)\psi_{n-1}(z) - f_n(z,x_0)\psi_n(z)\psi_{n-1}(x_0)] && (4.1)\\ B_n(z) &= \frac{1}{E_n}[f_n(x_0,z)\psi_n(x_0)\varphi_{n-1}(z) - f_n(z,x_0)\varphi_n(z)\psi_{n-1}(x_0)] && (4.2)\\ C_n(z) &= \frac{1}{E_n}[f_n(x_0,z)\varphi_n(x_0)\psi_{n-1}(z) - f_n(z,x_0)\psi_n(z)\varphi_{n-1}(x_0)] && (4.3)\\ D_n(z) &= \frac{1}{E_n}[f_n(x_0,z)\varphi_n(x_0)\varphi_{n-1}(z) - f_n(z,x_0)\varphi_n(z)\varphi_{n-1}(x_0)]. && (4.4) \end{aligned}$$

Here E_n is a real constant, see [2], [9, Ch. 11.3].

By *Christoffel-Darboux type formulas* (see e.g. [9, Ch. 11.3]) these functions can also be written in the form

$$\begin{aligned} A_n(z) &= (x_0 - z)\left[1 + \sum_{k=1}^{n-1} \psi_k(x_0)\psi_k(z)\right] && (4.5)\\ B_n(z) &= (x_0 - z)\left[-\frac{1 + x_0 z}{z - x_0} + \sum_{k=1}^{n-1} \psi_k(x_0)\varphi_k(z)\right] && (4.6)\\ C_n(z) &= (x_0 - z)\left[\frac{1 + x_0 z}{z - x_0} + \sum_{k=1}^{n-1} \varphi_k(x_0)\psi_k(z)\right] && (4.7)\\ D_n(z) &= (x_0 - z)\left[1 + \sum_{k=1}^{n-1} \varphi_k(x_0)\varphi_k(z)\right]. && (4.8) \end{aligned}$$

Remark 4.2. These definitions differ from those of [2] and [12] by a factor zx_0. This is done in order to avoid an irrelevant pole at the origin and instead place a pole at infinity. This is consistent with the fact that integrability of the constant functions impose one condition at infinity on the solutions of the moment problem. (Recall that in (3.1) π_0 corresponds to $\alpha_0 = \infty$, which is systematically made use of in [9].)

The results below follow from somewhat more general results in [2].

Theorem 4.3. *The functions A_n, B_n, C_n, D_n converge locally uniformly in $\mathbb{C} \setminus \Gamma$ to holomorphic functions A, B, C, D with simple pole at ∞ and essential singularities at the points of Γ. They are given by*

$$\begin{aligned} A(z) &= (x_0 - z)\left[1 + \sum_{k=1}^{\infty} \psi_k(x_0)\psi_k(z)\right] && (4.9)\\ B(z) &= (x_0 - z)\left[-\frac{1 + x_0 z}{z - x_0} + \sum_{k=1}^{\infty} \psi_k(x_0)\varphi_k(z)\right] && (4.10)\\ C(z) &= (x_0 - z)\left[\frac{1 + x_0 z}{z - x_0} + \sum_{k=1}^{\infty} \varphi_k(x_0)\psi_k(z)\right] && (4.11)\\ D(z) &= (x_0 - z)\left[1 + \sum_{k=1}^{\infty} \varphi_k(x_0)\varphi_k(z)\right]. && (4.12) \end{aligned}$$

PROOF. Follows from [2, Prop. 12]. □

Theorem 4.4. *The formula*

$$\Omega_\mu(z) = -\frac{A(z)g(z) - C(z)}{B(z)g(z) - D(z)} \tag{4.13}$$

establishes a one-to-one correspondence between all Pick functions g and all solutions μ of the indeterminate moment problem.

PROOF. Follows from [2, Thm. 9]. $\square$

Remark 4.5. In [2] and [12] the convergence result in Theorem 4.3 is formulated only for $z \in \mathbb{C} \setminus (\Gamma \cup \{\mathrm{i}\} \cup \{-\mathrm{i}\})$. However, the argument builds on the convergence results for $\sum_{k=0}^{\infty} |\varphi_k(z)|^2$ and $\sum_{k=0}^{\infty} |\psi_k(z)|^2$ discussed at the beginning of this section, which, as stated there, holds for $z \in \hat{\mathbb{C}} \setminus \Gamma$.

The following result is proved in [12].

Theorem 4.6. *Let $\alpha \in \Gamma$ and let V_α be a disk with center at α containing no other point of Γ. Then for every positive ε there exists a constant $M(\varepsilon)$ such that*

$$|F(z)| \le M(\varepsilon) \exp\left\{ \frac{\varepsilon}{|z-\alpha|} \right\}$$

for all $z \in V_\alpha \setminus \{\alpha\}$, where F is any of the functions A, B, C, D.

PROOF. This is [12, Thm. 4.4]. $\square$

Now consider an entire function Φ, and define $M(\Phi, r) = \max_{|z|=r} |\Phi(z)|$. We recall that the *order* $\rho(\Phi)$ of Φ is defined as

$$\rho(\Phi) = \inf\{\lambda : M(\Phi, r) \le \exp\{r^\lambda\} \text{ for sufficiently large } r\},$$

and the *type* $\sigma(\Phi)$ of Φ is defined as

$$\sigma(\Phi) = \inf\{s : M(\Phi, r) \le \exp\{s r^{\rho(\Phi)}\} \text{ for sufficiently large } r\}.$$

See [5, Ch. 2], [20, Ch. 9]. We shall introduce analogous concepts for the functions $F \in \{A, B, C, D\}$ (meaning for any holomorphic function F with a finite number of singularities).

Let Ψ be a function which is holomorphic in a deleted neighborhood $V_\alpha \setminus \{\alpha\}$ of a point α, and with a non-removable singularity at α. Set $M_\alpha(\Psi, r) = \max_{|z-\alpha|=r} |\Psi(z)|$. We define the *order* $\rho_\alpha(\Psi)$ *of* Ψ *at* α as

$$\rho_\alpha(\Psi) = \inf\left\{ \lambda : M_\alpha(\Psi, r) \le \exp\{r^{-\lambda}\} \text{ for sufficiently small } r \right\}$$

and the *type* $\sigma_\alpha(\Psi)$ *of* Ψ *at* α as

$$\sigma_\alpha(\Psi) = \inf\left\{ s : M_\alpha(\Psi, r) \le \exp\{s r^{-\rho_\alpha(\Psi)}\} \text{ for sufficiently small } r \right\}$$

Let $\gamma_p \in \Gamma$ and let F be any of the functions A, B, C, D. We shall write $M_p(F, r)$ for $M_{\gamma_p}(F, r)$, $\rho_p(F)$ for $\rho_{\gamma_p}(F)$ and $\sigma_p(F)$ for $\sigma_{\gamma_p}(F)$. Thus

$$\rho_p(F) = \inf\{\lambda : M_p(F, r) \le \exp\{r^{-\lambda}\} \text{ for sufficiently small } r\}$$

and

$$\sigma_p(F) = \inf\{s : M_p(F, r) \le \exp\{s r^{-\rho_p(F)}\} \text{ for sufficiently small } r\}.$$

Theorem 4.7. *Let $F \in \{A, B, C, D\}$ and $\gamma_p \in \Gamma$. Then*

$$(i)\ \ \rho_p(F) < 1 \quad \text{or} \quad (ii)\ \ \rho_p(F) = 1 \ \text{ and } \ \sigma_p(F) = 0.$$

PROOF. This is a rewriting of Theorem 4.6. $\square$

5. Zeros of the functions A,B,C,D

The quotient $-A/B$ is obtained from (4.13) for $g(z) \equiv \infty$ and $-C/D$ is obtained for $g(z) \equiv 0$. Consequently there exist two solutions μ_∞ and μ_0 of the moment problem such that

$$\frac{A(z)}{B(z)} = -\Omega_{\mu_\infty}(z) \quad \text{and} \quad \frac{C(z)}{D(z)} = -\Omega_{\mu_0}(z) \tag{5.1}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$.

The functions B_n and D_n are quasi-orthogonal with respect to the solutions of the moment problem and hence have simple real zeros. Then also B and D have only real zeros by Hurwitz' theorem (see e.g. [20, p. 49]). The zeros are isolated since B and D are holomorphic in $\mathbb{C} \setminus \Gamma$ with essential singularities at the points of Γ and simple poles at ∞. It follows that outside Γ the quotients A/B and C/D have only poles as singularities, these occurring among the zeros of B and D. The poles are simple since $-A/B$ and $-C/D$ are Pick functions by (5.1).

Proposition 5.1. *The support of μ_∞ consists of Γ and the poles of A/B, the support of μ_0 consists of Γ and the poles of C/D. At the poles of A/B and C/D, the corresponding measures have positive mass, while the points of Γ have zero mass. Every point in Γ is an accumulation point for poles of A/B and of C/D.*

PROOF. According to Theorem 4.1 the set Γ is contained in the support of all solutions of the moment problem. It follows from the *Perron-Stieltjes inversion formula* (see e.g. [1, p. 124]) that at each pole of A/B the measure μ_∞ has a mass point with value like the residuum at the pole, which is positive since $-A/B$ is a Pick function. At all points where A/B is holomorphic, the measure μ_∞ has mass zero. Similarly for μ_0.

Since the functions in $\mathscr{L}$ are integrable with respect to μ_∞ and μ_0, each point in Γ has μ_∞-measure and μ_0-measure equal to zero. From this and the fact already mentioned that Γ is contained in $\operatorname{supp}\mu_\infty$ and $\operatorname{supp}\mu_0$, every point of Γ must be an accumulation point for mass points in $\operatorname{supp}\mu_\infty$ and in $\operatorname{supp}\mu_0$. □

Proposition 5.2. *All the zeros of A,B,C,D are real.*

PROOF. We have already seen that the zeros of B and D are real. Since $-B/A$ and $-D/C$ are Pick functions and hence are holomorphic outside $\mathbb{R}$ and all the zeros of B and D are real, it follows that A and C are different from zero outside $\mathbb{R}$. □

Proposition 5.3.
a) A and B have no common zeros
b) C and D have no common zeros
c) A and C have no common zeros
d) B and D have no common zeros

PROOF. We find by calculation from the definitions (4.1-4.4) and use of the determinant formula (cf. e.g. [9, Ch. 11.2]) that

$$A_n(z)D_n(z) - B_n(z)C_n(z) = (1+z^2)(1+x_0^2)$$

for $z \notin \Gamma$. Hence also

$$A(z)D(z) - B(z)C(z) = (1+z^2)(1+x_0^2) \tag{5.2}$$

for $z \notin \Gamma$. Possible common zeros are real by Proposition 5.2. Thus $z = \pm \mathrm{i}$ are not common zeros, and the result follows from (5.2). □

Proposition 5.4. *All the zeros of A,B,C,D are simple.*

PROOF. This follows from Proposition 5.3 together with the fact that $-A/B$, B/A, $-C/D$ and D/C are Pick functions and hence have simple poles. □

Proposition 5.5. *a) Between two consecutive zeros of B there is exactly one zero of A, and vice versa.*

b) Between two consecutive zeros of D there is exactly one zero of C, and vice versa.

PROOF. a) Let $\{x_k\}_{k=1}^{\infty}$ denote a numbering of the zeros of B, or equivalently the poles of A/B. We then have

$$-\frac{A(z)}{B(z)} = \Omega_{\mu_\infty}(z) = \sum_{k=1}^{\infty} \mu_\infty(\{x_k\}) \frac{1+x_k z}{x_k - z},$$

where $\mu_\infty(\{x_k\}) > 0$ for all k by Proposition5.1. Near x_j the term $\frac{1+x_j z}{x_j - z}$ dominates in the series.

Let ξ and η be two consecutive zeros of B, $\xi < \eta$. We then have

$$\lim_{x\to\xi^+}\left(-\frac{A(x)}{B(x)}\right) = -\infty \quad \text{and} \quad \lim_{x\to\eta^-}\left(-\frac{A(x)}{B(x)}\right) = +\infty.$$

Hence by the intermediate value theorem, there is at least one value $\zeta \in (\xi,\eta)$ such that $A(\zeta)/B(\zeta) = 0$, and consequently $A(\zeta) = 0$.

Since B/A is a Pick function, there exists by *Herglotz-Riesz representation theorem* (see e.g. [1, p. 91]) a real constant a, a positive constant b and a (positive) measure ν_∞ such that

$$\frac{B(z)}{A(z)} = a + bz + \Omega_{\nu_\infty}(z).$$

As in the case of μ_∞, the support of ν_∞ consists of Γ and the poles of B/A, i.e., the zeros of A. Let now $\{y_k\}_{k=1}^{\infty}$ denote a numbering of these zeros. Then

$$\frac{B(z)}{A(z)} = a + bz + \int_\Gamma \frac{1+tz}{t-z} \mathrm{d}\nu_\infty(t) = a + bz + \sum_{k=1}^{\infty} \nu_\infty(\{y_k\}) \frac{1+y_k z}{y_k - z},$$

where $\nu_\infty(\{y_k\}) > 0$ for all k. In the same way as above, we conclude that between two consecutive zeros of A there is at least one zero of B.

From these results the statement of a) follows.

b) The argument is completely analogous to the argument under a). □

Proposition 5.6. *Between two consecutive zeros of B there is exactly one zero of D and vice versa.*

PROOF. Using the definitions (4.1-4.4) we find by direct calculation

$$\begin{aligned} B_n(z)D_n(\zeta) - B_n(\zeta)D_n(z) \quad = \quad & E_n^{-2} f_n(x_0,x_0)[\psi_n(x_0)\varphi_{n-1}(x_0) - \psi_{n-1}(x_0)\varphi_n(x_0)] \\ & \cdot [f_n(z,\zeta)\varphi_n(z)\varphi_{n-1}(\zeta) - f_n(\zeta,z)\varphi_{n-1}(z)\varphi_n(\zeta)]. \end{aligned}$$

By using the determinant formula (recall e.g. [9, Ch. 11.2]) on the first brackets to the right and the Christoffel-Darboux formula (recall e.g. [9, Ch. 11.3]) on the last brackets, we obtain

$$B_n(z)D_n(\zeta) - B_n(\zeta)D_n(z) = (1+x_0^2)(z-\zeta)\left[1 + \sum_{k=1}^{n-1} \varphi_k(z)\varphi_k(\zeta)\right].$$

Hence

$$B(z)D(\zeta) - B(\zeta)D(z) = (1+x_0^2)(z-\zeta)\left[1 + \sum_{k=1}^{\infty} \varphi_k(z)\varphi_k(\zeta)\right]. \tag{5.3}$$

Differentiation of (5.3) with respect to ζ for $\zeta = z$ gives

$$B(z)D'(z) - B'(z)D(z) = -(1+x_0^2)\left[1 + \sum_{k=1}^{\infty} \varphi_k(z)^2\right]. \tag{5.4}$$

The right-hand side of this formula is negative for all real z.

Let ξ and η be two consecutive zeros of B, $\xi < \eta$. Then $B'(\xi)$ and $B'(\eta)$ have opposite sign by Proposition 5.4. Consequently $D(\xi)$ and $D(\eta)$ have opposite sign by (5.4). From the intermediate value theorem it then follows that there is at least one zero ζ of D in (ξ, η).

In exactly the same way we conclude from (5.4) that between two consecutive zeros of D there is at least one zero of B.

From these results the statement of the proposition follows. □

Let Φ be an entire function with a sequence $\{z_k\}_{k=1}^{\infty}$ of zeros, such that $|z_k| \geq \delta > 0$ and ordered such that $\{|z_k|\}$ tends non-decreasingly to infinity. We recall that the *convergence exponent* $\tau(\Phi)$ of Φ is defined as

$$\tau(\Phi) = \inf \left\{ t \in \mathbb{R} : \sum_{k=1}^{\infty} \frac{1}{|z_k|^t} < \infty \right\}$$

and the *genus* $\kappa(\Phi)$ of Φ is defined as

$$\kappa(\Phi) = \max \left\{ t \in \mathbb{Z} : \sum_{k=1}^{\infty} \frac{1}{|z_k|^t} = \infty \right\}.$$

See [5, Ch. 2]. [20, Ch. 10].

Now let Ψ be a function which is holomorphic in a deleted neighborhood $V_\alpha \setminus \{\alpha\}$ of an essential singularity α. Assume there are infinitely many zeros of Ψ in $V_\alpha \setminus \{\alpha\}$, and let $\{z_k\}_{k=1}^{\infty}$ be a numbering of these zeros, ordered such that $\{|z_k - \alpha|\}$ is non-increasing. In analogy with the concepts above we define the *convergence exponent* $\tau_\alpha(\Psi)$ *of* Ψ *at* α as

$$\tau_\alpha(\Psi) = \inf \left\{ t \in \mathbb{R} : \sum_{k=1}^{\infty} |z_k - \alpha|^t < \infty \right\}$$

and the *genus* $\kappa_\alpha(\Psi)$ *of* Ψ *at* α as

$$\kappa_\alpha(\Psi) = \max \left\{ t \in \mathbb{Z} : \sum_{k=1}^{\infty} |z_k - \alpha|^t = \infty \right\}.$$

(These definitions are clearly independent of the neighborhood V_α as long as V_α contains no other singularities than α.)

Let F denote any of the functions A, B, C, D and let $\{z_{p,j}\}_{j=1}^{\infty}$ denote the zeros of F in a neighborhood of $\gamma_p \in \Gamma$, chosen such that every zero of F occurs exactly once as a $z_{p,j}$, ordered such that $\{|z_{p,j} - \gamma_p|\}_j$ is non-increasing. We shall write $\tau_p(F)$ and $\kappa_p(F)$ for $\tau_{\gamma_p}(F)$ and $\kappa_{\gamma_p}(F)$. Thus

$$\tau_p(F) = \inf \left\{ t \in \mathbb{R} : \sum_{j=1}^{\infty} |z_{p,j} - \gamma_p|^t < \infty \right\} \tag{5.5}$$

and

$$\kappa_p(F) = \max \left\{ t \in \mathbb{Z} : \sum_{j=1}^{\infty} |z_{p,j} - \gamma_p|^t = \infty \right\}. \tag{5.6}$$

(These definitions are clearly independent of the exact partition of the sequence of zeros of F in subsequences $\{z_{p,j}\}_j$.)

Theorem 5.7. *For each $\gamma_p \in \Gamma$ the following equalities hold:*

$$\tau_p(A) = \tau_p(B) = \tau_p(C) = \tau_p(D)$$

$$\kappa_p(A) = \kappa_p(B) = \kappa_p(C) = \kappa_p(D).$$

PROOF. This result follows immediately from the definitions (5.5-5.6) and Propositions 5.5-5.6. □

6. Factorization of the functions A,B,C,D

Let $\{\zeta_j\}_{j=1}^{\infty}$ be a sequence in $\mathbb{C}$, $\zeta_j \neq 0$ for all j, such that $\{|\zeta_j|\}$ tends monotonically to infinity. The *Weierstrass product* determined by this sequence is the expression

$$\Phi(\zeta) = \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{\zeta_j}\right) \exp\left\{\frac{\zeta}{\zeta_1} + \frac{\zeta^2}{2\zeta_2^2} + \cdots + \frac{\zeta^j}{j\zeta_j^j}\right\}.$$

This product converges locally uniformly in $\mathbb{C}$, and thus Φ represents an entire function with zeros exactly at the points ζ_j. See e.g. [5, Ch. 20], [20, Ch. 10].

Now let F denote any of the functions A,B,C,D. Let the zeros different from 0 and ∞ be partitioned in groups $\{z_{p,j}\}_{j=1}^{\infty}$ as described in Section 5. Recall that then $|z_{p,j} - \gamma_p| \to 0$ non-increasingly, and every zero of F (except possibly 0 and ∞) belongs to exactly one of these subsequences.

Let $\zeta = \frac{1}{z-\gamma_p}$, $\zeta_{p,j} = \frac{1}{z_{p,j}-\gamma_p}$. Then $\zeta \to \infty$ as $z \to \gamma_p$ and $\zeta_{p,j} \xrightarrow{j} \infty$. Since the Weierstrass product

$$S_p^{\infty}(\zeta) = \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{\zeta_{p,j}}\right) \exp\left\{\frac{\zeta}{\zeta_{p,1}} + \frac{\zeta^2}{2\zeta_{p,2}^2} + \cdots + \frac{\zeta^j}{j\zeta_{p,j}^j}\right\}$$

represents an entire function with zeros at $\{\zeta_{p,j}\}_j$, the function

$$S_p(z) = S_p^{\infty}\left(\frac{1}{z-\gamma_p}\right) \tag{6.1}$$

represents a function which is holomorphic in $\hat{\mathbb{C}} \setminus \{\gamma_p\}$ with zeros at the points $z_{p,j}$, $j = 1,2,\ldots$. Note that if 0 belongs to one of the sequences $\{\zeta_{p,j}\}_j$, then the product (6.1) has a factor $z/(z-\gamma_p)$. We shall call this function a *Weierstrass product at γ_p*.

In the proposition below, F denotes as before any of the functions A,B,C,D.

Proposition 6.1. *The function F can be factorized as*

$$F(z) = R(z) \prod_{p=1}^{q} S_p(z) T_p(z).$$

R is a rational function with all poles and zeros in the set Γ except for a simple pole at ∞, S_p is holomorphic in $\hat{\mathbb{C}} \setminus \{\gamma_p\}$ defined by the Weierstrass product (6.1), and T_p is holomorphic in $\hat{\mathbb{C}} \setminus \{\gamma_p\}$ without zeros.

PROOF. The argument here is essentially a modification of arguments found in [13, Sections 65–67].

We first assume that $F(0) \neq 0$. Then we define

$$f(z) = \frac{F(z)}{\prod_{p=1}^{q} S_p(z)}. \tag{6.2}$$

This function is holomorphic in $\mathbb{C} \setminus \Gamma$ with a simple pole at ∞ and without zeros.

At $z = \infty$ we have

$$\begin{aligned} f(z) &= uz + v + \frac{w}{z} + \cdots \\ f'(z) &= u - \frac{w}{z^2} - \cdots \\ \frac{f'(z)}{f(z)} &= \frac{1}{z} + \frac{s}{z^2} + \cdots. \end{aligned}$$

Thus f'/f is holomorphic in $\hat{\mathbb{C}} \setminus \Gamma$ and with a simple zero at ∞.

For every $\gamma_p \in \Gamma$ there is a Laurent series expansion of f'/f around γ_p. Let

$$h_p(z) = \frac{a_{-1}^{(p)}}{z-\gamma_p} + \sum_{k=2}^{\infty} \frac{a_{-k}^{(p)}}{(z-\gamma_p)^k}$$

denote the principal part of this series. We may then write

$$h_p(z) = \frac{a_{-1}^{(p)}}{z-\gamma_p} + g_p'(z), \quad g_p(z) = -\sum_{k=2}^{\infty} \frac{a_{-k}^{(p)}}{(k-1)(z-\gamma_p)^{k-1}}. \tag{6.3}$$

Note that h_p represents a holomorphic function in $\hat{\mathbb{C}} \setminus \{\gamma_p\}$. The difference $\frac{f'(z)}{f(z)} - \sum_{p=1}^{q} h_p(z)$ is thus holomorphic in all of $\hat{\mathbb{C}}$, and is consequently a constant. Thus

$$\frac{f'(z)}{f(z)} = b + \sum_{p=1}^{q} \frac{a_{-1}^{(p)}}{z-\gamma_p} + \sum_{p=1}^{q} g_p'(z). \tag{6.4}$$

By integrating along a small circle around γ_p we see that only the integral of $\frac{f'(z)}{f(z)}$ and of $\frac{a_{-1}^{(p)}}{z-\gamma_p}$ contributes to the value. The integral of $\frac{f'(z)}{f(z)}$ is determined up to a multiple of $2\pi\mathrm{i}$. It follows that the same is the case for the integral of $\frac{a_{-1}^{(p)}}{z-\gamma_p}$, and hence $a_{-1}^{(p)}$ is an integer. The behavior at infinity (cf. (6.3)) implies that $b = 0$ and $\sum_{p=1}^{q} a_{-1}^{(p)} = 1$. Thus

$$\frac{f'(z)}{f(z)} = \sum_{p=1}^{q} \frac{a_{-1}^{(p)}}{z-\gamma_p} + \sum_{p=1}^{q} g_p'(z) \quad \text{with} \quad \sum_{p=1}^{q} a_{-1}^{(p)} = 1. \tag{6.5}$$

Integration gives

$$\log f(z) = \sum_{p=1}^{q} a_{-1}^{(p)} \log(z-\gamma_p) + \sum_{p=1}^{q} g_p(z) + C$$

and by exponentiation we then obtain

$$f(z) = \mathrm{e}^C \prod_{p=1}^{q} (z-\gamma_p)^{a_{-1}^{(p)}} \cdot \prod_{p=1}^{q} \mathrm{e}^{g_p(z)}.$$

From (6.2) this may be written as

$$F(z) = R(z) \prod_{p=1}^{q} S_p(z) T_p(z) \tag{6.6}$$

where $S_p(z)$ denotes the Weierstrass product at γ_p determined by the sequence $\{z_{p,j}\}_j$, $T_p(z)$ denotes the holomorphic function $\mathrm{e}^{g_p(z)}$ in $\hat{\mathbb{C}} \setminus \{\gamma_p\}$, which is without zeros, and $R(z)$ denotes the rational function $e^C \prod_{p=1}^{q} (z-\gamma_p)^{a_{-1}^{(p)}}$. Because of (6.5), $R(z)$ has a simple pole at ∞.

Now assume that $F(0) = 0$, then the proof goes along the same lines with only minor modifications. We now set

$$f(z) = \frac{F(z)}{z \prod_{p=1}^{q} S_p(z)},$$

so that it is still holomorphic in $\mathbb{C} \setminus \Gamma$ without zeros and a simple pole at ∞. Again f'/f is holomorphic in $\hat{\mathbb{C}} \setminus \Gamma$, however the zero at ∞ is not simple but double.

This implies that in the formula (6.4) for f'/f, not only $b = 0$, but also $\sum_{p=1}^{q} a_{-1}^{(p)} = 0$.

Integration and exponentiation results in (6.6), where now $R(z) = ze^C \prod_{p=1}^{q} (z-\gamma_p)^{a_{-1}^{(p)}}$, but because $\sum_{p=1}^{q} a_{-1}^{(p)} = 0$, this is again a rational function with a simple pole at ∞ as claimed in the Proposition.

$\square$

We introduce the function F_p by

$$F_p(z) = S_p(z)T_p(z).$$

For a fixed p we again consider the transformation $z \to \zeta = \frac{1}{z-\gamma_p}$, $\zeta_{p,j} = \frac{1}{z_{p,j}-\gamma_p}$.

We define

$$S_p^\infty(\zeta) = S_p(z), \quad T_p^\infty(\zeta) = T_p(z), \quad \text{and} \quad F_p^\infty(\zeta) = F_p(z).$$

These are entire functions. S_p^∞ is a (classical) Weierstrass product, T_p^∞ has no zeros, and $F_p^\infty(\zeta) = S_p^\infty(\zeta)T_p^\infty(\zeta)$.

We recall the definitions of $\rho(\Phi)$, $\rho_p(\Psi)$, $\sigma(\Phi)$, $\sigma_p(\Psi)$, $\tau(\Phi)$, $\tau_p(\Psi)$, $\kappa(\phi)$ and $\kappa_p(\Psi)$ from Setions 4–5.

Proposition 6.2. *The following equalities hold:*

$$\begin{aligned} \rho_p(F) &= \rho_p(F_p) = \rho(F_p^\infty) && (6.7)\\ \sigma_p(F) &= \sigma_p(F_p) = \sigma(F_p^\infty) && (6.8)\\ \tau_p(F) &= \tau_p(F_p) = \tau(F_p^\infty) && (6.9)\\ \kappa_p(F) &= \kappa_p(F_p) = \kappa(F_p^\infty) && (6.10)\end{aligned}$$

PROOF. This follows directly from the definitions and the fact that the values of the rational function R and of the functions F_r for $r \neq p$ (which are holomorphic at γ_p) have no effect in the definitions. □

Let $\{\zeta_j\}_{j=1}^\infty$ be a sequence of points in $\mathbb{C}$ such that $\{|\zeta_j|\}$ tends non-decreasingly to infinity. Assume that there is a largest natural number κ such that $\sum_{j=1}^\infty \frac{1}{|\zeta_j|^\kappa}$ diverges. Then the infinite product

$$\Phi(\zeta) = \prod_{j=1}^\infty (1-\frac{\zeta}{\zeta_j})\exp\left\{\frac{\zeta}{\zeta_j} + \frac{\zeta^2}{2\zeta_j^1} + \cdots + \frac{\zeta^\kappa}{\kappa\zeta_j^\kappa}\right\}$$

converges locally uniformly in $\mathbb{C}$ and represents an entire function. See e.g. [5, Ch. 2], [20, Ch. 20]. Such products are called *canonical products* or *Hadamard products*. With $\zeta = \frac{1}{z-\gamma_p}$, the function $\Psi(z) = \Phi(\zeta) = \Phi(\frac{1}{z-\gamma_p})$ is then holomorphic in $\hat{\mathbb{C}} \setminus \{\gamma_p\}$. Such products are called *canonical products at γ_p* or *Hadamard products at γ_p*. See e.g., [5, Ch. 2], [20, Ch. 10].

Theorem 6.3. *Let F be any of the functions A, B, C, D. Then it can be decomposed in the following way:*

$$F(z) = R(z)\prod_{p=1}^q P_p(z)Q_p(z). \tag{6.11}$$

Here $R(z)$ is a rational function with all zeros and poles in the set Γ except for a simple pole at ∞, $P_p(z)$ is a canonical product at γ_p determined by the zeros $\{z_{p,j}\}_j$ and $Q_p(z)$ is a function holomorphic in $\hat{\mathbb{C}} \setminus \{\gamma_p\}$ without zeros.

PROOF. If follows from (6.7), Proposition 6.2 and Theorem 4.7 that $\rho(F_p^\infty) \le 1$. From the classical theory of entire functions of finite order it follows that $\tau(F_p^\infty) \le \rho(F_p^\infty)$ (see e.g. [5, Ch. 2], [20, Ch. 10]), hence in our case $\kappa(F_p^\infty) \in \{0,1\}$. Let P_p^∞ denote the canonical product determined by the sequence $\{\zeta_{p,j}\}_j = \{\frac{1}{z_{p,j}-\gamma_p}\}$. I.e.,

$$P_p^\infty(\zeta) = \prod_{j=1}^\infty (1-\frac{\zeta}{\zeta_{p,j}})\exp\left\{\frac{\zeta}{\zeta_{p,j}} + \frac{\zeta^2}{2\zeta_{p,j}^2} + \cdots + \frac{\zeta^\kappa}{\kappa\zeta_{p,j}^\kappa}\right\}$$

where $\kappa = \kappa(F_p^\infty)$. The function

$$Q_p^\infty(\zeta) = \frac{F_p^\infty(\zeta)}{P_p^\infty(\zeta)} \tag{6.12}$$

is then an entire function without zeros.

Set $P_p(z) = P_p^\infty(\zeta)$, $Q_p(z) = Q_p^\infty(\zeta)$ with $\zeta = \frac{1}{z-\gamma_p}$. Then by (6.12) and Proposition 6.1 we conclude that $F(z)$ is of the form (6.11) where R, P_p and Q_p have the stated properties. □

7. Equality of orders

From the classical theory of entire functions refered to above it follows that when an entire function Φ has finite order $\rho(\Phi)$, then

$$\Phi(\zeta) = P(\zeta)\exp\{q(\zeta)\}$$

where P is a canonical product and q is a polynomial of degree at most $\rho(\Phi)$. Furthermore,

$$\rho(\Phi) = \max\{\tau(\Phi),\ \deg q\}. \tag{7.1}$$

See e.g., [5, Ch. 2], [20, Ch. 10]. Thus in our case, $Q_p^\infty(\zeta) = \exp\{q^\infty(\zeta)\}$, with $\deg q^\infty \le 1$.

Proposition 7.1. *For each $\gamma_p \in \Gamma$, the following equality holds:*

$$\rho(F_p^\infty) = \tau(F_p^\infty).$$

PROOF. The proof follows closely the argument in [4]. For the sake of completeness we wish to present the argument here. Note that the function F_p^∞ does not arise as a function in a Nevanlinna matrix for a classical moment problem and thus the result does not follow from [4] directly. However the argument in [4] uses only properties that we know F_p^∞ to have.

We know that $\rho(F_p^\infty) \le 1$

(a) $\underline{\rho(F_p^\infty) = 0}$. Clearly $\sum_{j=1}^\infty \frac{1}{|\zeta_{p,j}|^t} = \infty$ for $t \le 0$, hence $\tau(F_p^\infty) \ge 0$, and thus $\rho(F_p^\infty) = \tau(F_p^\infty)$ by (7.1).

(b) $\underline{0 < \rho(F_p^\infty) < 1}$. From (7.1) follows that $\deg q = 0$ and $\rho(F_p^\infty) = \tau(F_p^\infty)$, since $\deg q$ is an integer.

(c) $\underline{\rho(F_p^\infty) = 1}$

(i) $\underline{\kappa(F_p^\infty) = 1}$. Then by the definition of $\kappa(F_p^\infty)$ we see that $\sum_{p=1}^\infty \frac{1}{|\zeta_{p,j}|} = \infty$, hence $\tau(F_p^\infty) \ge 1$. Then from (7.1) follows that $\rho(F_p^\infty) = \tau(F_p^\infty)$.

(ii) $\underline{\kappa(F_p^\infty) = 0}$. Since $\rho(F_p^\infty)$ is an integer and $\sigma(F_p^\infty) = 0$ by Theorem 4.7 and (6.8), a theorem of *Lindelöf* (see e.g. [5, Ch. 9.2]) implies that $\deg q^\infty = \rho(F_p^\infty) - 1 = 0$. Thus by (7.1) $\rho(F_p^\infty) = \tau(F_p^\infty)$. □

Theorem 7.2. *Consider an indeterminate rational moment problem with a finite set γ of singularities, all singularities of infinite order. Then for each $\gamma_p \in \Gamma$ the following equalities hold:*

$$\rho_p(A) = \rho_p(B) = \rho_p(C) = \rho_p(D).$$

PROOF. This follows immediately from Theorem 5.7, Proposition 6.2 and Proposition 7.1. □

8. References

[1] N.I. Akhiezer. *The classical moment problem and some related questions in analysis*. Hafner, New York, 1965.

[2] A. Almendral Vazquez. Nevanlinna parametrization of the solutions of some rational moment problems. *Analysis*, 23(2):107–124, 2003.

[3] C. Berg. Indeterminate moment problems and the theory of entire functions. *J. Comput. Appl. Math.*, 65:27–55, 1995.

[4] C. Berg and H.L. Pedersen. On the order and type of entire functions associated with an indeterminate Hamburger moment problem. *Ark. Math.*, 32:1–11, 1994.

[5] R.P. Boas. *Entire functions*. Academic Press, 1954.

[6] H. Buchwalter and C. Cassier. La paraméterization de Nevanlinna dans le problème des moments de Hamburger. *Expo. Math.*, 2:155–178, 1984.

[7] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions with poles on the unit circle. *J. Math. Anal. Appl.*, 182:221–243, 1994.

[8] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. A rational moment problem on the unit circle. *Methods Appl. Anal.*, 4(3):283–310, 1997.

[9] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. *Orthogonal rational functions*, volume 5 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, 1999.

[10] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonal rational functions on the real half line with poles in $[-\infty,0]$. *J. Comput. Appl. Math.*, 179(1-2):121–155, 2005.

[11] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. An indeterminate rational moment problem and Carathéodory functions. *J. Comput. Appl. Math.*, 219(2):359–369, 2008.

[12] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Growth properties of Nevanlinna matrices for rational moment problems. *J. Approx. Theory*, 162(12):2184–2201, 2010.

[13] A.R. Forsyth. *Theory of functions of a comples variables, vol. I.* Dover Publications, 1965.

[14] G. Freud. *Orthogonal polynomials*. Pergamon Press, Oxford, 1971.

[15] H. Hamburger. Über eine Erweiterung des Stieltjesschen Momentsproblem I. *Math. Ann.*, 81(2):235–319, 1920.

[16] H. Hamburger. Über eine Erweiterung des Stieltjesschen Momentsproblem II. *Math. Ann.*, 82(1):120–164, 1920.

[17] H. Hamburger. Über eine Erweiterung des Stieltjesschen Momentsproblem III. *Math. Ann.*, 82(3):168–187, 1921.

[18] W.B. Jones and O. Njåstad. Orthogonal Laurent polynomials and strong moment theory. *J. Comput. Appl. Math.*, 105:51–91, 1999.

[19] W.B. Jones, O. Njåstad, and W.J. Thron. Orthogonal Laurent polynomials and the strong Hamburger moment problem. *J. Math. Anal. Appl.*, 98:528–554, 1984.

[20] A. Markushevich. *Theory of functions of a complex variable*, volume 2. Chelsea Publishing Co., New York, 1967.

[21] R. Nevanlinna. Asymptotische Entwickelungen beschränkter Funktionen und das Stieltjessche Momentenproblem. *Ann. Acad. Sci. Fenn. Ser. A.*, 18(5):53pp., 1922.

[22] R. Nevanlinna. Über beschränkte analytische Funktionen. *Ann. Acad. Sci. Fenn. Ser. A.*, 32(7):75pp., 1929.

[23] O. Njåstad. Solutions of the strong Hamburger moment problem. *J. Math. Anal. Appl.*, 197:227–248, 1996.

[24] O. Njåstad. Orders of Nevanlinna matrices for strong moment problems. *East J. Approx.*, 13:467–481, 2007.

[25] O. Njåstad. Analytic functions associated with strong Hamburger moment problems. *Proc. Edinburgh Math. Soc.*, 52:181–187, 2009.

[26] M. Riesz. Sur le problème des moments I. *Ark. Mat. Astr. Fys.*, 16(12):1–21, 1922.

[27] M. Riesz. Sur le problème des moments II. *Ark. Mat. Astr. Fys.*, 16(19):1–23, 1922.

[28] M. Riesz. Sur le problème des moments III. *Ark. Mat. Astr. Fys.*, 17(16):1–52, 1923.

[29] J.A. Shohat and J.D. Tamarkin. *The problem of moments*, volume 1 of *Math. Surveys*. Amer. Math. Soc., Providence, R.I., 1943.

[30] M.H. Stone. *Linear transformations in Hilbert space and their applications to analysis*, volume 15 of *American Math. Soc. Coll. Publ.* Amer. Math. Soc., New York, 1932.